(Q) Calculate all first order partial derivatives and the directional derivative $f'(x;u)$ for each of the real valued functions defined on $\mathbb{R}^n$ as follows

(a) $f(x) = a \cdot x$, where a is a fixed vector in $\mathbb{R}^n$.

(b) $f(x) = \|x\|^4$

(c) $f(x) = x \cdot L(x)$, where $L : \mathbb{R}^n \to \mathbb{R}^n$ a linear function.

(d) $f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$, where $a_{ij} = a_{ji}$.

Solution (a) :→ Let $a = (a_1, \ldots, a_n)$

$x = (x_1, \ldots, x_n)$. Thus

$$f(x) = a \cdot x = a_1 x_1 + \cdots + a_n x_n \quad — (1)$$

Then we have

$$D_k f(x) = \frac{\partial f}{\partial x_k}(x) = a_k = a \cdot e_k;$$

$$k = 1, 2, \ldots, n.$$

$D_k f(x)$ is constant for all $k \Rightarrow D_k f(x)$ is continuous for all k.

~~THEOREM~~

$\therefore$ f is differentiable and hence directional derivative at any direction exists. Then,

$$f'(x;u) = \lim_{h\to 0} \frac{f(x+hu) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{a\cdot(x+hu) - a\cdot x}{h}$$

$$= \lim_{h\to 0} \frac{a\cdot(hu)}{h}$$

$$= \lim_{h\to 0} \frac{h\sum_{i=1}^{n} a_i u_i}{h}$$

$$= \sum_{i=1}^{n} a_i u_i = a\cdot u$$

(b) Solution :→

$$\because f(x) = \|x\|^4 = \left(\sum_{i=1}^{n} x_i^2\right)^2$$

$$= \sum_{i=1}^{n} x_i^4 + \sum_{i\neq j} x_i^2\cdot x_j^2$$

$$= x_k^4 + \sum_{i\neq k} x_i^4 + x_k^2\left(\sum_{i\neq k} x_i^2\right) + \left(\sum_{i\neq k} x_i^2\right) x_k^2$$

$$+ \sum_{i\neq k,\, j\neq k,\, i\neq j} x_i^2 x_j^2$$

$$= x_k^4 + \sum_{i\neq k} x_i^4 + 2x_k^2\left(\sum_{i\neq k} x_i^2\right) + \sum_{i\neq k,\, j\neq k,\, i\neq j} x_i^2 x_j^2$$

Let $k \in \{1, 2, \cdots, n\}$. Then.

~~Hence~~,

$$D_k f(x) = 4x_k^3 + 4x_k\left(\sum_{i\neq k} x_i^2\right)$$

$$= 4x_k\left(\sum_{i=1}^{n} x_i^2\right) = 4x_k \|x\|^2$$

Thus, $D_k f(x)$ exists & is continuous for all $k \in \{1, 2, ---, n\}$.

$\therefore$ f is differentiable, and hence directional derivative exists in all directions.

Let

$u = (u_1, ---, u_n) = u_1 e_1 + ----- + u_n e_n$.

Now

$$f'(x;u) = f'(x)u$$

$$= f'(x)(u_1 e_1 + -------- + u_n e_n)$$

$$= \sum_{k=1}^{n} u_k f'(x)(e_k)$$

$$= \sum_{k=1}^{n} u_k f'(x; e_k)$$

$$= \sum_{k=1}^{n} u_k \frac{\partial f}{\partial x_k}(x)$$

$$= \sum_{k=1}^{n} u_k (4x_k \|x\|^2)$$

$$= 4\|x\|^2 \sum_{k=1}^{n} x_k u_k$$

$$= 4\|x\|^2 (x \cdot u)$$

(c) Solution : $\rightarrow$

$$D_k f(x) = \lim_{h \to 0} \frac{f(x+he_k) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+he_k) \cdot L(x+he_k) - x \cdot L(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+he_k) \cdot (L(x) + L(he_k)) - x \cdot L(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x \cdot L(x)} + he_k \cdot L(x) + x \cdot L(he_k) + he_k L(he_k) - \cancel{x \cdot L(x)}}{h}$$

$$= \lim_{h \to 0} \frac{h e_k \cdot L(x) + h x \cdot L(e_k) + h^2 e_k \cdot L(e_k)}{h}$$

$$= \lim_{h \to 0} \left\{ e_k L(x) + x \cdot L(e_k) + h e_k \cdot L(e_k) \right\}$$

$$= e_k \cdot L(x) + x \cdot L(e_k)$$

By continuity of x and $L(x)$, we conclude that $D_k f(x)$ is continuous for all $k = 1, 2, \ldots, n$.

$\therefore f$ is differentiable & hence directional derivative of f exists in all directions.

Let $u = (u_1, \ldots, u_n)$. Then

$$f'(x;u) = \sum_{k=1}^{n} u_k D_k f(x)$$

$$= \sum_{k=1}^{n} u_k \left\{ x \cdot L(e_k) + e_k \cdot L(x) \right\}$$

$$= \sum_{k=1}^{n} \left(x \cdot u_k L(e_k) + u_k e_k \cdot L(x) \right)$$

$$= \sum_{k=1}^{n} \left(x \cdot L(u_k e_k) + u_k e_k \cdot L(x) \right)$$

$$= x \cdot \sum_{k=1}^{n} L(u_k e_k) + \sum_{k=1}^{n} (u_k e_k) \cdot L(x)$$

$$= x \cdot L\left(\sum_{k=1}^{n} u_k e_k \right) + \sum_{k=1}^{n} (u_k e_k) \cdot L(x)$$

$$= x \cdot L(u) + u \cdot L(x)$$

Answer.

solution (d) : →

$$\because \quad f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$$

$$= a_{kk} x_k^2 + \sum_{i \neq k} a_{ik} x_i x_k + \sum_{i \neq k} a_{ki} x_k x_i$$

$$+ \sum_{i \neq k,\, j \neq k,\, i \neq j} a_{ij} x_i x_j$$

Then we have

$$D_k f(x) = 2a_{kk}x_k + \sum_{i\neq k} a_{ik}x_i + \sum_{i\neq k} a_{ki}x_i$$

$$= 2a_{kk}x_k + 2\sum_{i\neq k} a_{ik}x_i \quad (\because a_{ik}=a_{ki})$$

$$= 2\sum_{i=1}^{n} a_{ik}x_i$$

Thus, $D_k f(x)$ exists and is continuous. Hence, f is differentiable and thus, it has directional derivative in every direction. Then

$$f'(x;u) = \sum_{k=1}^{n} u_k \frac{\partial f}{\partial x_k}(x)$$

$$= 2\sum_{i=1}^{n} u_k \left(\sum_{i=1}^{n} a_{ik}x_i\right)$$

$$= 2\sum_{i=1}^{n}\sum_{i=1}^{n} a_{ik}x_i u_k$$

$$= 2x^T A u, \quad \text{where } A=(a_{ij})_{i=1}^{n}{}_{j=1}^{n}$$